Lecture 20 04/08/2019

Radiation

Simple Radiating Systems

Consider a localized source oscillating at frequency ω:

$$\rho(\vec{x},t) = \rho(\vec{x})\, e^{-i\omega t} \quad , \quad \vec{J}(\vec{x},t) = \vec{J}(\vec{x})\, e^{-i\omega t}$$

In the Lorenz gauge (which is good for localized sources), we have:

$$\left(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)\vec{A} = -\mu_0 \vec{J}$$

Then (since there are no sources at ∞):

$$\vec{A}(\vec{x},t) = \frac{\mu_0}{4\pi}\int \vec{J}(\vec{x}\,',t')\, \frac{\delta\left(t - t' - \frac{|\vec{x}-\vec{x}\,'|}{c}\right)}{|\vec{x}-\vec{x}\,'|}\, d^3x'\, dt'$$

$\leftarrow$ retarded Green's function

Integration over t' results in:

$$\vec{A}(\vec{x},t) = \frac{\mu_0}{4\pi}\int \frac{\vec{J}\left(\vec{x}\,', t - \frac{|\vec{x}-\vec{x}\,'|}{c}\right)}{|\vec{x}-\vec{x}\,'|}\, d^3x'$$

After using $\vec{J}(\vec{x},t) = \vec{J}(\vec{x})\, e^{-i\omega t}$, we find:

$$\vec{A}(\vec{x},t) = \vec{A}(\vec{x})\, e^{-i\omega t}$$

Where:

$$\vec{A}(\vec{x}) = \frac{\mu_0}{4\pi} \int \vec{J}(\vec{x}') \frac{e^{i\frac{\omega}{c}|\vec{x}-\vec{x}'|}}{|\vec{x}-\vec{x}'|} d^3x'$$

We consider the long wavelength limit where the source dimension "d" is small compared to the wavelength $\lambda \equiv \frac{2\pi c}{\omega}$. The three possible cases then are:

(a) Near field region ($d \ll \lambda$, $r \ll \lambda$). In this case, $\frac{\omega}{c}|\vec{x}-\vec{x}'| \ll 2\pi$, and hence:

$$\vec{A}(\vec{x}) \simeq \frac{\mu_0}{4\pi} \int \frac{\vec{J}(\vec{x}')}{|\vec{x}-\vec{x}'|} d^3x'$$

This is called the "static limit", in which (apart from the $e^{-i\omega t}$ factor) $\vec{A}$ is the same as that for a steady current. Therefore, in this case, magnetostatics holds.

(b) Far field region ($d \ll \lambda$, $r \gg d, \lambda$). In this case, we have:

$$\vec{A}(\vec{x}) \simeq \frac{\mu_0}{4\pi} \int \frac{\vec{J}(\vec{x}')}{r} e^{ik\sqrt{r^2 + r'^2 - 2\vec{x}\cdot\vec{x}'}} d^3x' \simeq \frac{\mu_0}{4\pi} \frac{e^{ikr}}{r} \int \vec{J}(\vec{x}') e^{-ik\frac{\vec{x}}{r}\cdot\vec{x}'} d^3x' \Rightarrow A(\vec{x}) \simeq \frac{\mu_0}{4\pi} \frac{e^{ikr}}{r} \int \vec{J}(\vec{x}') \, e^{-ik\hat{n}\cdot\vec{x}'} d^3x'$$

$$\hat{n} \equiv \frac{\vec{x}}{r}$$

The integral is essentially the Fourier transform of the current density.

(c) Arbitrary observation distance. In this case, we use:

$$\frac{e^{ik|\vec{x}-\vec{x}'|}}{|\vec{x}-\vec{x}'|} = ik \sum_{l,m} j_l(kr_<)\, h_l^{(1)}(kr_>)\, Y_{lm}(\theta,\phi)\, Y_{lm}^*(\theta',\phi')$$

Here, $j_l(x) = (-x)^l \left(\frac{1}{x}\frac{d}{dx}\right)^l \left(\frac{\sin x}{x}\right)$ are the spherical Bessel functions.

Also:

$$h_l^{(1)}(x) = j_l(x) + i n_l(x)$$

Where:

$$n_l(x) = (-x)^l \left(\frac{1}{x}\frac{d}{dx}\right)^l \left(\frac{-\cos x}{x}\right)$$

For $\vec{x}$ sufficiently outside the source, $r_< = r'$ and $r_> = r$. Hence:

$$\vec{A}(\vec{x}) = \mu_0\, ik \sum_{l,m} h_l^{(1)}(kr)\, Y_{lm}(\theta,\phi) \left[\int \vec{J}(\vec{x}')\, j_l(kr')\, Y_{lm}^*(\theta',\phi')\, d^3x'\right]$$

In the long wavelength limit, $kr' \ll 2\pi$ and $j_l(kr') \simeq \frac{(kr')^l}{(2l+1)!!}$,

where $(2l+1)!! = (2l+1)(2l-1)\ldots 3 \cdot 1$.

Keeping only the first term $l=0$, then, we have $j_0(kr') \simeq 1$ and

$h^{(1)}(kr) = \frac{e^{ikr}}{ikr}$. This results in:

$$\vec{A}(\vec{x}) \simeq \frac{\mu_0}{4\pi}\frac{e^{ikr}}{r}\int \vec{J}(\vec{x}')\, d^3x' = \frac{\mu_0}{4\pi}\frac{e^{ikr}}{r}\int \overset{0}{\cancel{\vec{\nabla}'\cdot(\vec{x}'\vec{J})}}\, d^3x'$$

(localized source)

$$-\frac{\mu_0}{4\pi}\frac{e^{ikr}}{r}\int \vec{x}'\, \vec{\nabla}'\cdot\vec{J}\, d^3x'$$

Unlike magnetostatic, where $\vec{\nabla}'\cdot\vec{J}(\vec{x}')=0$, we now have (from conservation of electric charge):

$$\vec{\nabla}'\cdot\vec{J}(\vec{x}',t) + \frac{\partial \rho(\vec{x}',t)}{\partial t} = 0 \Rightarrow \vec{\nabla}'\cdot\vec{J}(\vec{x}') = i\omega\rho(\vec{x}')$$

Thus:

$$\vec{A}(\vec{x}) \simeq -i\omega\underbrace{\int \vec{x}'\rho(\vec{x}')\, d^3x'}_{\vec{p}} \Rightarrow \boxed{\vec{A}(\vec{x},t) = \frac{-i\mu_0\omega}{4\pi}\vec{p}\,\frac{e^{ikr}}{r}e^{-i\omega t}}$$

Here $\vec{p}e^{-i\omega t} = \int \vec{x}'\rho(\vec{x}',t)d^3x'$ is the oscillating electric dipole moment of the source.

Therefore:

$$\vec{B}(\vec{x},t) = \vec{\nabla}\times\vec{A}(\vec{x},t) \simeq \frac{-i\mu_0\omega}{4\pi}\vec{\nabla}\left(\frac{e^{ikr}}{r}\right)\times\vec{p}\,e^{-i\omega t} = \frac{\mu_0\omega k}{4\pi}\frac{e^{ikr}}{r}$$

$$\left(1-\frac{1}{ikr}\right)\hat{n}\times\vec{p}\,e^{-i\omega t}$$

Also, from Maxwell equations, we have:

$$-i\omega \vec{E}(\vec{x},t) = \frac{1}{\mu_0 \epsilon_0} \vec{\nabla} \times \vec{B}(\vec{x},t) \Rightarrow \vec{E}(\vec{x},t) = \frac{i}{\mu_0 \epsilon_0 \omega} \vec{\nabla} \times \vec{B}(\vec{x},t)$$

Hence:

$$\vec{E}(\vec{x},t) = \frac{ik}{4\pi} \vec{\nabla} \times \left[\frac{e^{ikr}}{r}\left(1 - \frac{1}{ikr}\right) \hat{n} \times \vec{p}\right] e^{-i\omega t}$$

Note that $\hat{n} \equiv \frac{\vec{x}}{r}$. Then:

$$\vec{\nabla} \times \left[\frac{e^{ikr}}{r}\left(1 - \frac{1}{ikr}\right) \hat{n} \times \vec{p}\right] = \vec{\nabla} \times \left[\frac{e^{ikr}}{r}\left(1 - \frac{1}{ikr}\right) \frac{\vec{x}}{r} \times \vec{p}\right] =$$

$$\vec{\nabla}\left[\frac{e^{ikr}}{r^2}\left(1 - \frac{1}{ikr}\right)\right] \times (\vec{x} \times \vec{p}) + \frac{e^{ikr}}{r^2}\left(1 - \frac{1}{ikr}\right) \vec{\nabla} \times (\vec{x} \times \vec{p}) =$$

$$\left[\frac{e^{ikr}}{r^2} ik\left(1 - \frac{1}{ikr}\right) + e^{ikr}\left(\frac{-2}{r^3} + \frac{3}{ikr^4}\right)\right] \hat{n} \times (\hat{n} \times \vec{p})\, r + \frac{e^{ikr}}{r^2}$$

$$\left(1 - \frac{1}{ikr}\right)\left[\underbrace{(\vec{p} \cdot \vec{\nabla})\vec{x}}_{\vec{p}} - \underbrace{(\vec{\nabla} \cdot \vec{x})}_{3}\vec{p}\right] = ik \frac{e^{ikr}}{r} \hat{n} \times (\hat{n} \times \vec{p}) + e^{ikr}$$

$$\left(\frac{-3}{r^2} + \frac{3}{ikr^3}\right) \hat{n} \times (\hat{n} \times \vec{p}) - 2\vec{p}\, e^{ikr}\left(\frac{1}{r^2} - \frac{1}{ikr^3}\right) = ik \frac{e^{ikr}}{r} \hat{n} \times (\hat{n} \times \vec{p})$$

$$- e^{ikr}\left(\frac{1}{r^2} - \frac{1}{ikr^3}\right)\left[3(\hat{n} \cdot \vec{p})\hat{n} - \vec{p}\right]$$

This leads to:

$$\vec{E}(\vec{x},t) = \left[\frac{-k^2}{4\pi\epsilon_0} \frac{e^{ikr}}{r} \hat{n} \times (\hat{n} \times \vec{p}) + \frac{1}{4\pi\epsilon_0}\left(3\hat{n}(\hat{n} \cdot \vec{p}) - \vec{p}\right)\left(\frac{1}{r^3} - \frac{ik}{r^2}\right)\right] e^{-i\omega t}$$

In the near zone (i.e., $kr \ll 1$), we have:

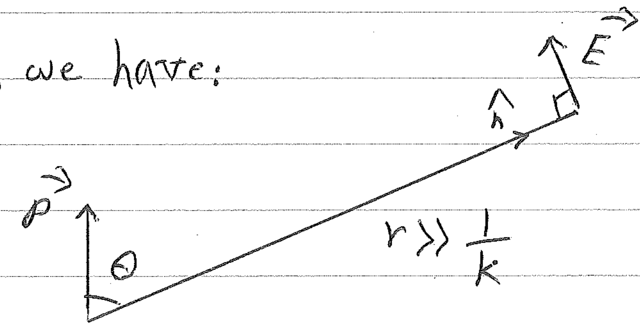
$$\vec{E}(\vec{x},t) \simeq \frac{1}{4\pi\varepsilon_0} \frac{3\hat{n}(\hat{n}\cdot\vec{p}) - \vec{p}}{r^3} e^{-i\omega t}$$

This is a quasi-static $\vec{E}$ field. In the far zone (i.e., $kr \gg 1$):

$$\vec{E}(\vec{x},t) \simeq \frac{k^2}{4\pi\varepsilon_0} \frac{e^{ikr}}{r} (\hat{n}\times\vec{p})\times\hat{n}$$

It is important to note that $\vec{E} \perp \hat{n}$ in the far zone, while $\vec{B} \perp \hat{n}$ in general (see the expression for $\vec{B}$ on page 4).

For a linearly polarized source, we have:



The time-averaged power per unit solid angle is given by:

$$\frac{dP}{d\Omega} = Re\left[\frac{1}{2}(\vec{E}\times\vec{H}^*)\cdot\hat{n}\right] r^2 = \frac{1}{2} Re\left[\vec{E}\cdot(\vec{H}^*\times\hat{n})\right] r^2 \Rightarrow \frac{dP}{d\Omega} =$$

$$\frac{1}{2} \frac{ck^4}{(4\pi)^2\varepsilon_0} \left|(\hat{n}\times\vec{p})\times\hat{n}\right|^2$$

For $\vec{p}$ real, we have:

$$\left|(\hat{n}\times\vec{p})\times\hat{n}\right| = \left|\hat{n}\times\vec{p}\right| = |\vec{p}| \sin\theta$$

Thus:

$$\frac{dP}{d\Omega} = \frac{ck^4}{32\pi^2\epsilon_0} |\vec{p}|^2 \sin^2\theta$$

The total radiated power is then given by:

$$P = \int \frac{ck^4}{32\pi^2\epsilon_0} |\vec{p}|^2 \sin^2\theta \, d\Omega = \frac{ck^4 |\vec{p}|^2}{12\pi\epsilon_0}$$

Some comments are in order at this point. First, the expression for the radiated power is valid for a general polarization when the components of $\vec{p}$ do not have the same phase. Second, the radiated power scales with the frequency $\propto \omega^4$. This is the Rayleigh scattering.

Example: A charge q oscillating at frequency ω along the z axis.

In this case, we have:

$z(t) = z_0 \sin\omega t$ ($kz_0 \ll 1$ in the long wavelength limit)

The physical electric dipole moment is:

$$\vec{p}_{phys} = q z(t) \hat{z} = q z_0 \sin\omega t \, \hat{z} = Re(i q z_0 \hat{z} e^{-i\omega t})$$

Hence:

$\vec{p} = i q z_0 \hat{z}$ (Linear polarization)

And:

$$\frac{dP}{d\Omega} = \frac{c k^4}{32\pi^2 \varepsilon_0} |\hat{n} \times \vec{p}\,|^2 = \frac{c k^4}{32\pi^2 \varepsilon_0} q^2 z_0^2$$

We note that in order to calculate $\vec{p}$ we can calculate $\vec{p}_{phys}$ first and then express it as the real part of $\vec{p}e^{-i\omega t}$.